ECE 346: Digital Signal Processing
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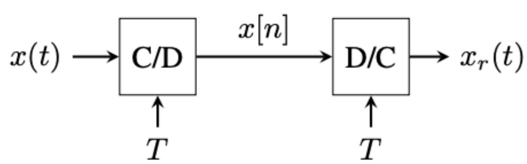
Spring 2021
Week of March 8, 2021

Recitation #6: Sampling Theory

Objective & Outline

- Problems $1-3$: recitation problems
- Problem 4: self-assessment problem

Problem 1 (Sampling)**.** Consider the following block diagram of a sampling system:



Suppose $T = 1/100$ seconds and let the continuous-time Fourier transform (CTFT) of $x(t)$ be defined as

$$X(j\Omega) = \begin{cases} \frac{\Omega^2}{100\pi^2}, & |\Omega| \leq 50\pi \\ 0.1, & 50\pi < |\Omega| \leq 120\pi, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

(a) Provide a labeled plot of $X(j\Omega)$.

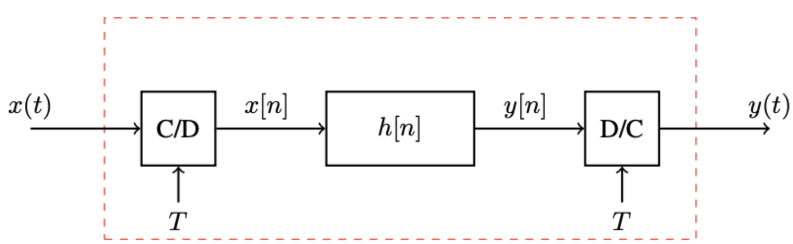
(b) Provide a labeled plot of the CTFT of the impulse sampled signal, $x_p(t)$, where $x_p(t)$ is defined as

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT). \quad (2)$$

(c) Provide a labeled plot of the DTFT of $x[n]$.

(d) Provide a labeled plot of the CTFT of $x_r(t)$, the reconstructed signal.

Problem 2 (Sampling)**.** Consider the following block diagram of a DSP system:



where $T = 1/500$ seconds and the input to this diagram is

$$x(t) = 2\sin(100\pi t) - \cos(300\pi t) \tag{3}$$

and $h[n]$ is an ideal low pass filter with cutoff frequency $\omega_c = \frac{2\pi}{5} \frac{\text{rad}}{\text{sec}}$.

(a) State and plot the continuous-time Fourier transform of $x(t)$, $X(j\Omega)$.

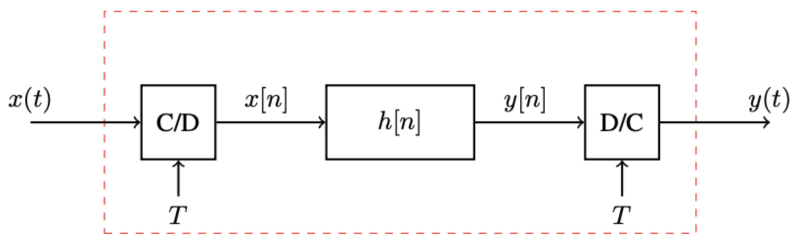
(b) State and plot the discrete-time Fourier transform of $x[n]$, $X(e^{j\omega})$.

(c) State and plot the discrete-time Fourier transform of $y[n]$, $Y(e^{j\omega})$.

(d) State and plot the continuous-time Fourier transform of $y(t)$, $Y(j\Omega)$.

(e) Does the "dashed block" act between the input and output as as a linear, time-invariant (LTI) system? Justify your answer.

Problem 3 (Sampling)**.** Consider the following block diagram of a digital signal processing system:



Suppose $T = 1/500$ seconds, while the frequency response of the discrete-time system $h[n]$ is given by

$$H(e^{j\omega}) = \begin{cases} 2e^{-j2\omega}, & |\omega| \le \frac{\pi}{8}, \\ 3, & \frac{\pi}{8} < |\omega| \le \frac{\pi}{4}, \\ 0, & \frac{\pi}{4} < |\omega| \le \pi. \end{cases} \tag{4}$$

Let the input signal $x(t)$ be defined by the expression:

$$x(t) = 3\sin(600\pi t) + 2\cos\left(100\pi t - \frac{\pi}{8}\right) - 5\cos\left(40\pi t - \frac{\pi}{6}\right). \tag{5}$$

(a) Provide a closed-form expression for the CTFT of $x(t)$.

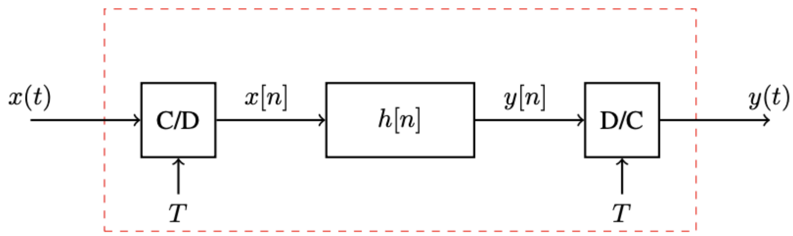
(b) Provide a closed-form expression for the DTFT of $x[n]$ for $|\omega| \le \pi$.

(c) Provide a closed-form expression for the DTFT of $y[n]$ for $|\omega| \le \pi$.

(d) Provide a closed-form expression for the CTFT of $y(t)$.

(e) Does the "dashed block" act between the input and output as as a linear, time-invariant (LTI) system? Justify your answer.

Problem 4 (Self-assessment)**.** Consider the following block diagram of a digital signal processing system:



Let $x(t)$ be defined by the following expression:

$$x(t) = 5\sin(500\pi t) + 7\cos(1000\pi t) + 3\cos(1800\pi t). \tag{6}$$

Suppose that there was an anti-aliasing block before the C/D block that produces an output $x_a(t)$. Let this anti-aliasing filter be an ideal low-pass filter with cutoff frequency 700 Hz.
Lastly, let $T = 1/900$ seconds and the discrete-time system $h[n]$ be defined as

$$h[n] = \frac{2\sin(\frac{2\pi}{3}n)}{\pi n} \tag{7}$$

Determine the following:

(a) Provide an expression for $x_a(t)$ and the CTFT of $x_a(t)$, $X_a(j\Omega)$.

(b) Provide an expression for $x[n]$ and provide a plot of its DTFT, $X(e^{j\omega})$.

(c) Provide labeled plots for the DTFT of $y[n]$, $Y(e^{j\omega})$ and the CTFT of the reconstructed signal, $Y(j\Omega)$.

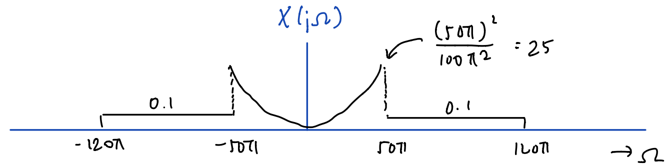
(d) Does the "dashed block" act between the input and output as as a linear, time-invariant (LTI) system? Justify your answer.

Problem 1 :

(a) We are given

$$X(j\Omega) = \begin{cases} \frac{\Omega^2}{100\pi^2} & , |\Omega| \le 50\pi \\ 0.1 & , 50\pi \le |\Omega| \le 120\pi \\ 0 & , \text{otherwise.} \end{cases}$$

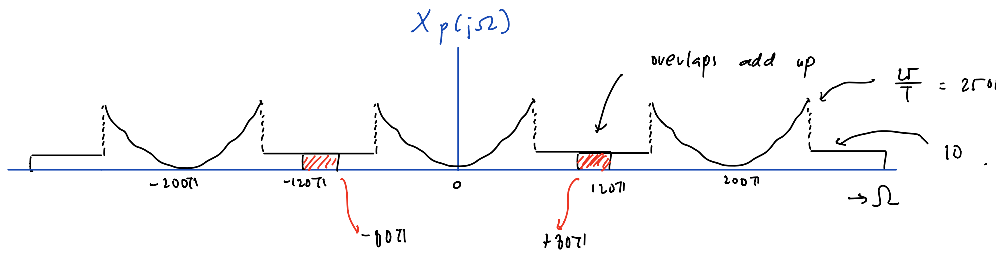
Since we have an Ω^2 factor from $|\Omega| \le 50\pi$, we should get the sense that the plot is a parabola of some sort. Building an easy intuition for small things like this is useful!

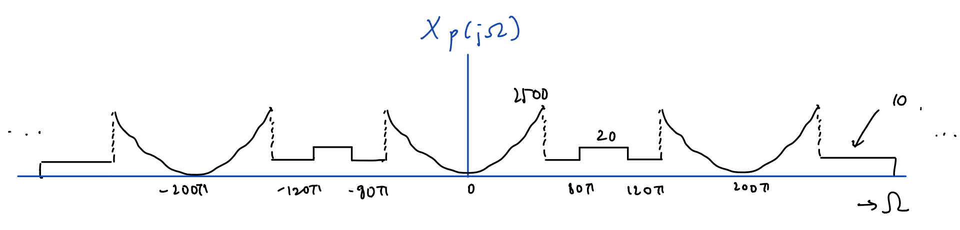


(b) We know the relationship

$$X_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - \frac{2\pi}{T}k)),$$

where $X_p(j\Omega)$ is the CTFT of $x_p(t)$. Note that $T = 1/100$ seconds and so our sampling frequency is $\Omega_s = 200\pi$ rads/sec.

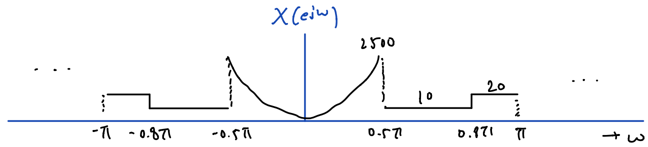




(c) We can get the DTFT $X(e^{j\omega})$ by looking $X_p(j\Omega)$ and using the relationship

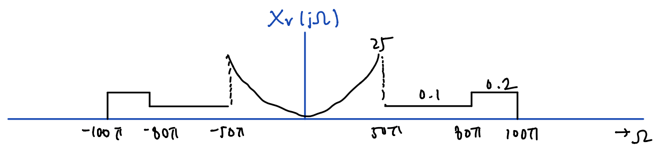
$$\omega = \Omega T.$$

Also note that since the DTFT frequencies are periodic $[-\pi, \pi]$, we can just look at there frequencies:



(d) Now we can take a look at $X(e^{j\omega})$ and use the relationship

$$\omega = \Omega / T .$$



Problem 2:

(a) We are given the input

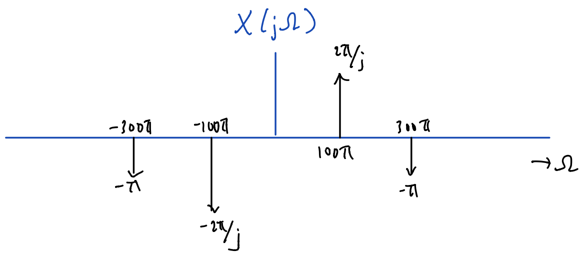
$$x(t) = 2\sin(100\pi t) - \cos(300\pi t).$$

Taking the CTFT of $x(t)$ yields

$$x(t) = \frac{1}{j} e^{j100\pi t} - \frac{1}{j} e^{-j100\pi t} - \frac{1}{2} e^{j300\pi t} - \frac{1}{2} e^{-j300\pi t}.$$

$$X(j\Omega) = \frac{2\pi}{j}\,\delta(\Omega - 100\pi) - \frac{2\pi}{j}\,\delta(\Omega + 100\pi) - \pi\,\delta(\Omega - 300\pi) - \pi\,\delta(\Omega + 300\pi).$$

The plot of this would be:

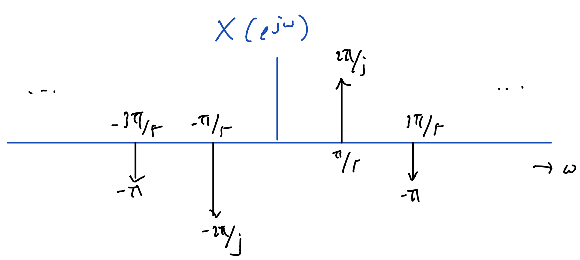


(b) We can directly look at $X(j\Omega)$ and scale the frequencies using the equation

$$\omega = \Omega T:$$

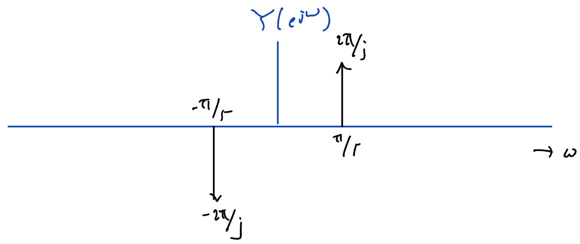
$$X(e^{j\omega}) = \frac{2\pi}{j}\,\delta(\omega - \pi/5) - \frac{2\pi}{j}\,\delta(\omega + \pi/5)$$
$$- \pi\,\delta(\omega - 3\pi/5) - \pi\,\delta(\omega + 3\pi/5).$$

Note that we implicitly used the property that $\delta(ax - x_0) = \frac{1}{|a|}\,\delta(x - x_0/a)$ and did NOT scale the amplitudes.



(c) The LPF filters out the signal with frequencies $\omega = \pm 3\pi/5$.
Thus,

$$Y(e^{j\omega}) = \frac{2\pi}{j}\,\delta(\omega - \pi/5) - \frac{2\pi}{j}\,\delta(\omega + \pi/5)$$



(d) To go from the DTFT to the CTFT, we can use the equation

$$\Omega = \omega / T.$$

Thus,

$$Y(j\Omega) = \frac{2\pi}{j}\,\delta(\Omega - 100\pi) - \frac{2\pi}{j}\,\delta(\Omega + 100\pi)$$

$$y(t) = 2\sin(100\pi t).$$

(e) Since no new frequencies were added, our dashed block is an LTI system, satisfying

$$Y(j\Omega) = X(j\Omega) \cdot H_{eff}(j\Omega),$$

where

$$H_{eff}(j\Omega) = \begin{cases} 1, & |\Omega| \le 200\pi \\ 0, & \text{otherwise}. \end{cases}$$

Problem 3 :

(a) We have the expression

$$x(t) = 3\sin(600\pi t) + 2\cos(100\pi t - \pi/8) - 5\cos(40\pi t - \pi/6).$$

Taking the CTFT yields

$$X(j\Omega) = 3\pi/j \; \delta(\Omega - 600\pi) - 3\pi/j \; \delta(\Omega + 600\pi) + 2\pi \delta(\Omega - 100\pi) e^{-j\pi/8}$$
$$+ 2\pi \delta(\Omega + 100\pi) e^{j\pi/8} - 5\pi \delta(\Omega - 40\pi) e^{-j\pi/6} - 5\pi \delta(\Omega + 40\pi) e^{j\pi/6}.$$

(b) To get $X(e^{j\omega})$, we can look at $X(j\Omega)$ and use the fact that

$$\omega = \Omega T.$$

$$X(e^{j\omega}) = 3\pi/j \; \delta(\omega - \tfrac{6}{5}\pi) - 3\pi/j \; \delta(\omega + \tfrac{6}{5}\pi) + 2\pi \delta(\omega - \tfrac{\pi}{5}) e^{-j\pi/8}$$
$$+ 2\pi \delta(\omega + \tfrac{\pi}{5}) e^{j\pi/8} - 5\pi \delta(\omega - \tfrac{2\pi}{25}) e^{-j\pi/6} - 5\pi \delta(\omega + \tfrac{2\pi}{25}) e^{j\pi/6}.$$

Since we want to look at discrete frequencies from $[-\pi, \pi]$:

$$\pm \tfrac{6}{5}\pi \text{ discrete frequencies} \rightarrow \mp \tfrac{4}{5}\pi .$$

Thus,

$$X(e^{j\omega}) = 3\pi/j \; \delta(\omega + \tfrac{4}{5}\pi) - 3\pi/j \; \delta(\omega - \tfrac{4}{5}\pi) + 2\pi \delta(\omega - \tfrac{\pi}{5}) e^{-j\pi/8}$$
$$+ 2\pi \delta(\omega + \tfrac{\pi}{5}) e^{j\pi/8} - 5\pi \delta(\omega - \tfrac{2\pi}{25}) e^{-j\pi/6} - 5\pi \delta(\omega + \tfrac{2\pi}{25}) e^{j\pi/6}.$$

(c) To obtain $Y(e^{j\omega})$, we have to look at the frequencies in $X(e^{j\omega})$ and shift & multiply correspondingly. These frequencies are

$$\tfrac{\pi}{4} \le \tfrac{4}{5}\pi \le \pi . \quad \text{(gets filtered)}$$
$$\tfrac{\pi}{8} \le \tfrac{\pi}{5} \le \tfrac{\pi}{4} \quad \text{(multiply by 3)}$$
$$\tfrac{2\pi}{25} \le \tfrac{\pi}{8} \quad \text{(multiply by 3, phase shift } e^{-j4\pi/25}\text{)}.$$

Thus,

$$Y(e^{j\omega}) = 6\pi \delta(\omega - \pi/5) e^{-j\pi/8} + 6\pi \delta(\omega + \pi/5) e^{j\pi/8} - 10\pi \delta(\omega - 2\pi/25) e^{-j\pi 49/150}$$
$$- 10\pi \delta(\omega + 2\pi/25) e^{j\pi 49/150}.$$

(d) Now using the relationship

$$\Omega = \frac{\omega}{T},$$

we get

$$Y(j\Omega) = 6\pi\delta(\Omega - 100\pi)e^{-j\pi/8} + 6\pi\delta(\Omega + 100\pi)e^{j\pi/8}$$
$$-10\pi\delta(\Omega - 40\pi)e^{-j\pi 49/150} - 10\pi\delta(\Omega + 40\pi)e^{j\pi 49/150}.$$

(e) We can see from (a) and (d) that

$$Y(j\Omega) = X(j\Omega)H_{eff}(j\Omega),$$

where

$$H_{eff}(j\Omega) = \begin{cases} 2e^{-j\Omega/250}, & |\Omega| \le 62.5\pi, \\ 3, & 62.5\pi < |\Omega| \le 125\pi, \\ 0, & \text{otherwise.} \end{cases}$$

$\boxtimes$

Problem 4 (Self- Assessment) :

$\boxed{(a)}$ $$X(j\Omega) = \frac{3\pi}{j}\delta(\Omega-600\pi) - \frac{3\pi}{j}\delta(\Omega+600\pi) + 2\pi\delta(\Omega-100\pi)e^{-j\frac{\pi}{8}} + 2\pi\delta(\Omega+100\pi)e^{j\pi/8} - 5\pi\delta(\Omega-40\pi)e^{-j\pi/6} - 5\pi\delta(\Omega+40\pi)e^{j\pi/6}$$

$\boxed{(b)}$ Using $\Omega = \frac{\omega}{T}$ and the property $\delta(ax-x_0) = \frac{1}{|a|}\delta(x-\frac{x_0}{a})$, we get :

$$X(e^{j\omega}) = \frac{3\pi}{j}\delta(\omega-\tfrac{6}{5}\pi) - \frac{3\pi}{j}\delta(\omega+\tfrac{6}{5}\pi) + 2\pi\delta(\omega-\tfrac{\pi}{5})e^{-j\pi/8} + 2\pi\delta(\omega+\tfrac{\pi}{5})e^{j\pi/8} - 5\pi\delta(\omega-\tfrac{2\pi}{25})e^{-j\pi/6} - 5\pi\delta(\omega+\tfrac{2\pi}{25})e^{j\pi/6}$$

$\pm\frac{6}{5}\pi$ in discrete frequencies $= \mp\frac{4\pi}{5}$

$$\Rightarrow X(e^{j\omega}) = \frac{3\pi}{j}\delta(\omega+\tfrac{4\pi}{5}) - \frac{3\pi}{j}\delta(\omega-\tfrac{4\pi}{5}) + 2\pi\delta(\omega-\tfrac{\pi}{5})e^{-j\pi/8} + 2\pi\delta(\omega+\pi/5)e^{j\pi/8} - 5\pi\delta(\omega-\tfrac{2\pi}{25})e^{-j\pi/6} - 5\pi\delta(\omega+\tfrac{2\pi}{25})e^{j\pi/6}$$

$\boxed{(c)}$ $$Y(e^{j\omega}) = 6\pi\delta(\omega-\tfrac{\pi}{5})e^{-j\pi/8} + 6\pi\delta(\omega+\pi/5)e^{j\pi/8} - 10\pi\delta(\omega-\tfrac{2\pi}{25})e^{-j\pi\frac{49}{150}} - 10\pi\delta(\omega+\tfrac{2\pi}{25})e^{j\pi\frac{49}{150}}$$

$\boxed{(d)}$ Using the same relation as in (b), we get:

$$Y(j\Omega) = 6\pi\delta(\Omega-100\pi)e^{-j\pi/8} + 6\pi\delta(\Omega+100\pi)e^{j\pi/8} - 10\pi\delta(\Omega-40\pi)e^{-j\pi\frac{49}{150}} - 10\pi\delta(\Omega-40\pi)e^{j\pi\frac{49}{150}}$$

$\boxed{(e)}$ We see from (a) and (d) that

$$Y(j\Omega) = X(j\Omega)H_{eff}(j\Omega)$$

where $$H_{eff}(j\Omega) = \begin{cases} 2e^{-j\frac{\Omega}{250}}, & |\Omega| \le 62.5\pi \\ 3, & |\Omega| \le 125\pi \\ 0, & \text{otherwise.} \end{cases}$$